

REVISITING A THEOREM OF L.A. SHEPP ON OPTIMAL STOPPING

PHILIP A. ERNST AND LARRY A. SHEPP

ABSTRACT. Using a bondholder who seeks to determine when to sell his bond as our motivating example, we revisit one of Larry Shepp's classical theorems on optimal stopping. We offer a novel proof of Theorem 1 from [7]. Our approach is that of guessing the optimal control function and proving its optimality with martingales. Without martingale theory one could hardly prove our guess to be correct.

1. Introduction

Consider a bondholder who must determine when to sell his bond. Each corporate or municipal zero coupon bond has a fixed face value at which the bond will be redeemed at some specific time b in the future and a value at which the bond is traded and which fluctuates from day to day. Suppose the current price or value is higher than the face value of the bond. Should the owner of the bond sell it or hold it for some time, with the hope that the bond will rise to an even higher value over the face value?

Brownian motion conditioned to have the current value at time 0 and the face value at time b is both a simple and mathematically appealing model for the fluctuations of the price of the bond during $[0, b]$ ¹ Since only the difference between the current and final values is important, we may, without loss of generality, assume that the price at time 0 is 0 and is $-a$ at the final time b . The bond holder would like to know for which values of $-a$ and b should he or she decide to sell his or her security.

Let $W_{a,b}^*(t)$ denote the ordinary Brownian motion process $W(t)$ *conditioned* so that $W(0) = 0$ and $W(b) = -a$. We want to choose a selling time, or optimal stopping time, τ to maximize $V^*(a, b) = \mathbb{E}[W^*(\tau)]$. Suppose we know V and believe in the model, and that $V(a_0(b), b) = 0$. The result is that if a bond is trading at a price more than $a_0(b)$ above its face value b at a time b before the termination date then we should immediately sell it.

Considering optimal stopping problems under a completely different guise (indeed, bond liquidation was far from the preoccupations of the time) the solution

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¹For now, we ignore the fact that this gives an unrealistic model (at least for zero coupon bonds) because these are observed *never* to trade at a price higher than face value. Most consumers would never buy a zero coupon bond if the price is higher than the face value. But, if they did, then these models would be useful in setting the optimal selling price.

to the maximization problem defined in the previous paragraph was solved in [7]! Specifically, in this paper, it is proved that the value of α in the expression $a_0(b) = \alpha\sqrt{b}$, is $\alpha = .83992\dots$. The proof of the latter, according to the authors of [7], was “incredibly dense” because the calculations were made without martingale theory². To appreciate how difficult it is to solve the bondholder’s problem without using martingale theory, one only need glance at the calculations in [7].

Using the bondholder as our motivating example, we proceed to offer a novel proof of the classical result in [7] by guessing the optimal control function and using martingale theory to prove its optimality. The reader is encouraged to consult [3], in which the motivation, reasoning, and success of this strategy is well documented.

2. Guessing the Optimal Control Function

We first wish to determine for which a and b the inequality $\mathbb{E}[W_{a,b}^*(\tau)] \leq 0$ will hold for all τ . The process $W_{a,b}^*(t)$ can be written in terms of the ordinary Wiener process $W(t)$ for $t \geq 0$ by the following simple formula

$$W_{a,b}^*(t) = -at/b + (1 - t/b)W(t/(1 - t/b)). \quad (2.1)$$

We justify equation (2.1) by first noting that the processes on opposite sides of the equality are the same process. This is because $\sqrt{b}W(t/\sqrt{b})$ is a Wiener process and the process W^* was defined in Section 1 as the Wiener process conditioned so that $W(1) = 0$ has zero mean and covariance $\min(s, t) - st$. Thus W^* is the same process as $(1 - t)W(t/(1 - t))$. This allows us to restate the question more simply in terms of W itself. Indeed, if we make the (monotonic) change of stopping time variable $\tau = t/(1 - t/b)$, then from (2.1) we have $t = \tau/(1 + \tau/b)$ and

$$W_{a,b}^*(t) = -a + b \frac{a + W(\tau)}{b + \tau}. \quad (2.2)$$

Since τ runs through all stopping times on $[0, \infty)$ as t runs over stopping times on $[0, b]$, we see that a, b is a pair satisfying $\mathbb{E}[W_{a,b}^*(t)] \leq 0$ if and only if for all stopping times τ of W , we have

$$V(a, b) := \frac{\mathbb{E}[a + W(\tau)]}{b + \tau} \leq \frac{a}{b}.$$

We now turn to the problem of finding $V(a, b)$ for all a, b . We begin our search by making a guess, following the same stochastic optimization approach as that of [3]. Intuitively, we should stop at $t = 0$ if a is sufficiently large compared to b , i.e., if $a \geq f(b)$. It is also clear that once $f(b)$ is determined, then the optimal stopping rule will be to stop the first time t that $a + W(t) \geq f(b + t)$. This is because the problem at time t is the same as at time 0, with simply a different value of (a, b) , namely $(a + W(t), b + t)$. The unknown function, $a = f(b)$, is the free boundary which we have to find. It is also clear that if we do quit at $t = 0$, i.e., $a \geq f(b)$,

²Personal communication

then $V(a, b) = a/b$. If $a < f(b)$ then we must continue to sample or observe the ratio, and we must have at some small time h that

$$V(a, b) \doteq \mathbb{E}[V(a + W(h), b + h)]. \quad (2.3)$$

We can either stop the process at time 0 or allow the process to run for a small time h and reassess. If we do the latter, then, assuming $V(a, b)$ is twice differentiable in a and once differentiable in b , we may expand V in a Taylor's series, and use Itô calculus to obtain

$$V(a, b) = \mathbb{E}[V(a, b) + V_1(a, b)W(h) + 1/2V_{11}(a, b)W^2(h) + V_2(a, b)h + o(h)]. \quad (2.4)$$

Since $\mathbb{E}[W(h)] = 0$ and $\mathbb{E}[W^2(h)] = h$, we can subtract $V(a, b)$ and divide by h and let $h \rightarrow 0$. Doing so, we obtain that if (a, b) is a point where we continue, then we must have the following partial differential equation:

$$0 = \frac{1}{V_{11}(a, b)} + V_2(a, b). \quad (2.5)$$

Again, recall that we are still only heuristically just trying to *guess* the right f . Since W has the property that the Brownian motion process scales quadratically in a rescaling of time, i.e., $W(t) \sim \sqrt{b}W'(t/b)$, where W' is another Brownian motion, we can write

$$V(a, b) = \sup_{\tau} \mathbb{E}\left[\frac{\frac{a}{b} + \frac{W(\tau)}{b}}{1 + \frac{\tau}{b}}\right] = \frac{1}{\sqrt{b}} \sup_{\tau} \frac{\frac{a}{\sqrt{b}} + W'(\tau)}{1 + \tau} = \frac{1}{\sqrt{b}} V\left(\frac{a}{\sqrt{b}}, 1\right), \quad (2.6)$$

as τ/b runs through all stopping times. This means that

$$V(a, b) = (1/\sqrt{bh})(a/\sqrt{b})$$

for some $h(x) = V(x, 1)$. If this is substituted into the partial differential equation above in (2.5), we get an ordinary differential equation for the function h . This brings us much closer to our solution. The ordinary differential equation for h is $h''(u) = uh'(u) + h(u)$. This ordinary differential equation has two linearly independent solutions, $h_1(u) = e^{1/2u^2}$ and $h_2(u) = \int_0^\infty e^{\lambda u - \lambda^2/2} d\lambda$. The latter expression is known as the parabolic cylinder function (the Whittaker function). Every solution of the ordinary differential equation must be a linear combination of these. The first solution does not look right for large u since it grows too fast. We discard it (again, it is our right to do so, as we are only guessing). The other solution is the one we want. The form of h suggests that the free boundary is $f(b) = c\sqrt{b}$, for some c . It remains to determine the constant multiplier of the second solution to the ordinary differential equation and the value of c . So far, we have the guess

$$V(a, b) = B \int_0^\infty e^{\lambda a - \lambda^2 b/2} d\lambda, a < c\sqrt{b}. \quad (2.7)$$

This must fit smoothly to a/b at the boundary $a = c\sqrt{b}$. This readily yields two equations (continuity of the zeroth and first derivatives) which uniquely determine the constants, B and c . This gives us the guess, which we denote \hat{V} , as

$$\hat{V}(a, b) = (1 - \alpha^2) \int_0^\infty e^{\lambda a - \lambda^2 b/2} d\lambda, \quad a < \alpha\sqrt{b}, \quad (2.8)$$

where $\alpha = .83992\dots$ is the unique root of the transcendental equation

$$\alpha = (1 - \alpha^2) \int_0^\infty e^{\lambda\alpha - 1/2\lambda^2} d\lambda. \quad (2.9)$$

3. Proving Our Guess is Correct

Theorem 3.1. *The \hat{V} above in equation (2.8) gives the correct answer, i.e., $V \equiv \hat{V}$.*

Proof. We use the supermartingale inequality to prove $V \leq \hat{V}$. Define the process $Y_t = \hat{V}(a + W_t, b + t)$ where \hat{V} is defined as in the above guess. One can easily check that Y is expectation-decreasing, $\mathbb{E}[dY_t] < 0$, if $a > \alpha\sqrt{b}$ and that $\mathbb{E}[dY_t] = 0$ if $a < \alpha\sqrt{b}$. Thus Y is always expectation-decreasing, $\mathbb{E}[dY_t] \leq 0$ and so for any stopping time τ we have $\mathbb{E}[Y_\tau] \leq \mathbb{E}[Y_0]$. It is also easy to check that $\hat{V}(a, b) \geq a/b$ for all a and b , and so we have, for any stopping time τ that

$$\mathbb{E}\left[\frac{a + W(\tau)}{b + \tau}\right] \leq \mathbb{E}\left[\hat{V}(a + W(\tau), b + \tau)\right] = \mathbb{E}[Y_\tau] \leq \mathbb{E}[Y_0] = \hat{V}(a, b) \quad (3.1)$$

By the definition of V this holds for every τ , and so $V(a, b) \leq \hat{V}(a, b)$, for all a and b . It is easy to see that for the stopping time τ defined as the first t for which $a + W(t) = \alpha\sqrt{b + t}$ that equality holds throughout. Since this is a legitimate stopping time, we have that $V \equiv \hat{V}$, completing the proof. \square

We thus conclude that the bond should be sold if and only if the current value is at least $\alpha\sqrt{b}$ where b is the time until redemption. This is a reasonable strategy which has been put into practice; see [2]. Further related literature includes [6], [4], and [5].

Remark 3.2. Another model for bond trading instead of the pinned Brownian motion model above would be the Black-Scholes model [1] used analogously as before, i.e., pinned exponential Brownian motion. This model has the advantage of never taking negative values, which conforms to the reality that bonds do not take on negative values. However, the (great) disadvantage is that it seems impossible to obtain the explicit fair value, i.e. to determine when the bond should be sold. Using (2.1), we see that the problem for pinned exponential Brownian motion amounts to finding

$$\sup_\tau \mathbb{E}\left[e^{-c\frac{a+W(\tau)}{b+\tau}}\right]. \quad (3.2)$$

It seems difficult to work with this model analytically, but a numerical solution should be valuable. A drawback of this model is that it allows the bond to trade higher than its face value, which is almost never observed in reality.

4. An Open Question

We conclude by offering the reader an open question. Consider the process \hat{W} obtained by pinning the Wiener process to a *random* point at time $t = 1$. Condition the Wiener process by choosing a random variable X and conditioning so that $W(1) = X$. Assuming that only the distribution of X is known, but the sample value is not (although more and more information about X is obtained by observing more and more of the path of \hat{W}), find $\sup_{\tau} \mathbb{E} [\hat{W}(\tau)]$. This is an unsolved and difficult problem; the above method of our paper fails and it seems unlikely that an explicit solution can be given.

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PHILIP A. ERNST: DEPARTMENT OF STATISTICS, RICE UNIVERSITY, HOUSTON, TX 77005, USA

E-mail address: philip.ernst@rice.edu

LARRY A. SHEPP (DECEASED APRIL 2013); DEPARTMENT OF STATISTICS, THE WHARTON SCHOOL OF THE UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PA 19104, USA